MA 227 (Calculus-III) Midterm test #1

Show your work. Each problem is 20 points Thu, Sep 9, 2004

1. Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position:

$$\mathbf{a}(t) = 2\mathbf{i} + 6t\,\mathbf{j}, \qquad \mathbf{v}(0) = -2\mathbf{i} + \mathbf{k}, \qquad \mathbf{r}(0) = \mathbf{j} + 5\,\mathbf{k}.$$

Answers:

$$\mathbf{v}(t) = (2t-2)\mathbf{i} + 3t^2\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{r}(t) = (t^2 - 2t)\mathbf{i} + (t^3 + 1)\mathbf{j} + (t+5)\mathbf{k}$$

2. The position of a moving particle is given by

$$\mathbf{r}(t) = \langle t,\, t^3,\, \ln t\rangle$$

(a) Find velocity vector, speed, and acceleration vector at the point where $t = 1$;
(b) Find the curvature of the particle's trajectory at the point where $t = 1$;
(c) Find the tangential and normal components of the acceleration vector at the point where $t = 1$.

Answers:

$$\mathbf{v}(1) = \langle 1, 3, 1\rangle, \qquad v(1) = \sqrt{11},$$
$$\mathbf{a}(1) = \langle 0, 6, -1\rangle$$

and

$$\kappa(1) = \frac{\sqrt{118}}{11^{3/2}}, \qquad a_T(1) = \frac{17}{\sqrt{11}}, \qquad a_N(1) = \frac{\sqrt{118}}{\sqrt{11}}$$

3. Find the length of the curve:

$$\mathbf{r}(t) = \langle \cos 3t,\ \sin 3t,\ 2(t-1)^{3/2} \rangle \qquad 1 \leq t \leq 4.$$

Answer:

$$L = \int_1^4 \sqrt{9\sin^2 3t + 9\cos^2 3t + 9(t-1)}\, dt = \int_1^4 3\sqrt{t}\, dt = 14.$$

4. A vector function is given:

$$\mathbf{r}(t) = 4t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}.$$

(a) Find the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the unit binormal vector $\mathbf{B}$.
(Bonus) Find equations of the normal plane and the osculating plane at the point where $t = \pi/2$.

Answer:

$$\begin{aligned}
\mathbf{T}(t) &= \langle \tfrac{4}{5}, -\tfrac{3}{5}\sin t, \tfrac{3}{5}\cos t\rangle \\
\mathbf{N}(t) &= \langle 0, -\cos t, -\sin t\rangle \\
\mathbf{B}(t) &= \langle \tfrac{3}{5}, \tfrac{4}{5}\sin t, -\tfrac{4}{5}\cos t\rangle
\end{aligned}$$

Normal plane:

$$4x - 3y = 8\pi$$

Osculating plane:

$$3x + 4y = 6\pi$$

5. Find parametric equations for the following surfaces:
(a) the hemisphere $x^2 + y^2 + z^2 = 1$, $x > 0$ (which lies in front of the plane $x = 0$);
(b) the surface obtained by rotating the curve $x^3 + 2y = 10$, $1 \le y \le 5$, about the x-axis.

Answers:
(a) either $x = \sqrt{1 - y^2 - z^2}$, or

$$\begin{aligned} x &= \cos\theta \sin\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\phi \end{aligned}$$

with $0 \le \phi \le \pi$ and $0 \le \theta \le \pi$ (it is important to specify the domain, since otherwise it will describe the entire sphere).
(b)

$$\begin{aligned} x &= x \\ y &= \frac{10 - x^3}{2} \cos\theta \\ z &= \frac{10 - x^3}{2} \sin\theta \end{aligned}$$

with $0 \le \theta \le 2\pi$ and $0 \le x \le 2$ (again, it is important to specify the domain, because otherwise the equations will describe a much longer curve).