ON A CONJECTURE OF ERDŐS, PACH, POLLACK AND TUZA

ÉVA CZABARKA, UNIVERSITY OF SOUTH CAROLINA

Moon (and independently several others) showed that for a fixed minimum degree $\delta \geq 2$, the diameter $\mathrm{diam}(G)$ of every connected graph G of order n satisfies $\mathrm{diam}(G) \leq \frac{3n}{\delta+1} + O(1)$, as $n \to \infty$. This upper bound is sharp (even for δ-regular graphs), but the constructions have complete subgraphs, whose order increases with n. In 1989 Erdős, Pach, Pollack, and Tuza conjectured that this upper bound can be improved, if large cliques are excluded (which we refer to as the EPPT conjecture). More precisely they conjectured that, if G contains no K_{k+1} (and a few technical conditions on k, δ, n are satisfied) then $\mathrm{diam}(G) \leq \frac{C_k n}{\delta} + O(1)$, as $n \to \infty$, where for even k, $C_k = 3 - \frac{2}{k}$, and for odd k, $C_k = 3 - \frac{2}{k}\left(1 + \frac{1}{(k+1)^2 - 2}\right)$. They proved the EPPT conjecture for triangle-free graphs ($k = 2$) and created examples showing that their conjecture, if true, is sharp.

Peter Dankelmann suggested that we strengthen the condition K_{k+1}-free to k-colorable. Note that the original constructions also show that this weaker conjecture also must be sharp. In 2009 Dankelmann and Székely and myself have shown that every connected 4-colorable graph G of order n and minimum degree $\delta \geq 1$ satisfies $\mathrm{diam}(G) \leq \frac{5n}{2\delta} - 1$, which is the weaker form of the EPPT conjecture for K_5-free graphs

In 2021 Singgih and Székely and myself gave an infinite family of graphs that yielded a counterexample of the EPPT conjecture for all odd values of k, even under the stronger condition of k-colorability. Our examples show that the smallest possible value for C_k in the EPPT conjecture is $C_k = 3 - \frac{2}{k}$ regardless of the parity of k. We also proved that for any $k \geq 3$, if G is a connected k-colorable graph of minimum degree at least $\delta \geq 2$, then $\mathrm{diam}(G) \leq \left(3 - \frac{1}{k-1}\right)\frac{n}{\delta} - 1$. We introduced a linear programming based approach to attack the weaker conjecture (with the colorability assumption).

Using the linear programming approach, Smith, Székely and myself recently showed that for $k \in \{3, 4\}$, any connected k-colorable graph G of order n, and of minimum degree at least $\delta \geq 2$ satisfies $\mathrm{diam}(G) \leq \left(3 - \frac{2}{k}\right)\frac{n}{\delta} - 1$. For $k = 4$ this provides a shorter and simpler proof of the theorem with Dankelmann.