# Joint Program Exam in Real Analysis

# May 5, 2020

**Instructions:**

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure unless noted otherwise. Instead of $d\mathrm{m}$ we sometimes write $dx$, $dt$, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order $p$ with respect to the positive measure $\mu$ and $\| \cdot \|_p$ denotes the norm on $L^p(X, \mu)$. We also use the abbreviation $L^p(I)$ for $L^p(I, \mathrm{m})$ when $I$ is a subinterval of $\mathbb{R}$.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f \in L^1(X, \mu) \cap L^\infty(X, \mu)$. Prove or disprove that $f \in L^p(X, \mu)$ for $1 < p < \infty$.

2. If $f \in L^1(\mathbb{R})$, compute
$$\lim_{n\to\infty} \int_0^\infty \frac{x^n}{1+x^n} f(x)dx.$$

3. Suppose that $g \in L^1(\mathbb{R})$ and $E \subset \mathbb{R}$ is a set of finite Lebesgue measure. Define
$$f(x) = \int_{x+E} g(y)\, dy.$$
Prove that $f \in L^1(\mathbb{R})$ and
$$\int_{\mathbb{R}} f\, dx = \mathrm{m}(E) \int_{\mathbb{R}} g\, dx.$$

4. Let $f(x) = x^2 \cos(1/x)$ when $x \in (0, 1]$ and $f(0) = 0$. Show that $f$ is absolutely continuous on $[0, 1]$.

5. Let $f : [0, 1] \to \mathbb{R}$ be absolutely continuous. If $E$ is a subset of $[0, 1]$ with $\mathrm{m}(E) = 0$ show that $\mathrm{m}(f(E)) = 0$.

6. Let $\varphi \in L^\infty([0, 1])$. Given an integrable function $f$ on $[0, 1]$ define $M_\varphi f = \varphi f$. Prove that $M_\varphi : L^1([0, 1]) \to L^1([0, 1])$ is bounded and that
$$\|M_\varphi\| = \|\varphi\|_\infty.$$
Recall that, if $B$ is a Banach space, a linear operator $T : B \to B$ is called bounded if there is a number $C$ such that $\|Tx\| \le C\|x\|$ for all $x \in B$. The norm $\|T\|$ of $T$ is then defined as the infimum over all such numbers $C$.

7. Let $f$ be Lebesgue integrable on $\mathbb{R}$. For $h > 0$ and $x \in \mathbb{R}$ define
$$\psi_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt.$$
   (a) Prove that $\|\psi_h\|_1 \le \|f\|_1$.
   (b) Show that $\lim_{h\downarrow 0} \|\psi_h - f\|_1 = 0$.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $f, f_k \in L^1(\mu)$ for each $k \in \mathbb{N}$. If $f_k \to f$ pointwise almost everywhere and if $\|f_k\| \to \|f\|$ as $k \to \infty$, show that
$$\lim_{k\to\infty} \int_X |f_k - f| d\mu = 0.$$